Game number of double generalized petersen graphs

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Abstract
A double generalised Petersen graph DP (n, m) is studied for its game chromatic number χg. We get the exact values of the game chromatic numbers for DP(n,1), DP(n,2), DP(n,3), and DP(n, m). The investigation of the game chromatic number of generalised petersen and jahangir graphs was previously conducted by Shaheen, Ramy, Ziad Kanaya, and Khaled Alshehada; these findings build upon their work.

Keywords: coloring of graphs; game chromatic number; double general- ized petersen graph.

1. Introduction

We consider the following well-known graph coloring game, played on a simple graph G with a color set C of cardinality k. Two players, Alice and Bob, alternately color an uncolored vertex of G with a color from C such that no adjacent vertices receive the same color (such a coloring of a graph G is known as proper coloring). Alice has the first turn and the game ends when no move is possible any more. If all the vertices are properly colored, Alice wins, otherwise Bob wins. The game chromatic number of G, denoted by χg(G), is the least cardinality k of the set C for which Alice has a winning strategy. In other words, necessary and sufficient conditions for k to be game chromatic number of a graph G are:

i) Bob has winning strategy for k − 1 colors or less and
ii) Alice has winning strategy for k colors.

This parameter is well defined, since it is easy to see that Alice always wins if the number of colors is larger than the maximum degree of G. Clearly, χg(G) is at least as large as the ordinary chromatic number χ(G), but it can be considerably more.

The obvious bounds for the game chromatic number are:

$$\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$$

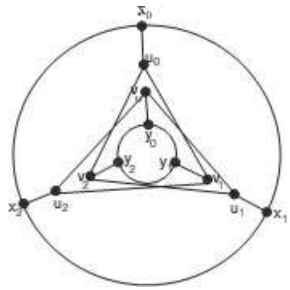
where χ(G) is the chromatic number and Δ(G) is the maximum degree of the graph G. A lot of attempts have been made to determine the game chromatic number for several classes of graphs. This work was initiated by Faigle et al. [6]. It was proved by Kierstead and Trotter [9] that the maximum of game chromatic number of a forest is 4, also that 33 is an upper bound for

game chromatic number of planar graphs. Bodlaender [5], found that the game chromatic number of Cartesian product is bounded above by constant in the
family of planar graph. Later, Bartnicki et al. [4] determine the exact values of χg(GQH) when G and H belong to certain classes of graphs, and show that, in general, the game chromatic number χg(GQH) is not bounded from above by a function of game chromatic numbers of graphs G and
H. After that, Zhu [12] established a bound for game coloring number and acyclic
chromatic number for Cartesian product of two graphs H and S. In [10], Sia determined the exact values for the Cartesian product of different families of graph like SmQPn, SmQCn, P2QWn. In [3], S. A. Bokhary and M. S. Akhtar found the the game chromatic number of some convex polytope graphs. Further game chromatic number of splitting graphs of path and cycle is determined by M. S. Akhtar et al. In [11] Shaheen, Ramy, Ziad Kanaya and Khaled Alshehada found the game chromatic number of generalized petersen graphs and jahangir graphs.
Here, we introduce some definitions and notational conventions. Sup- pose that Alice and Bob play the coloring game with k colors. We say that there is a threat to an uncolored vertex v if there are k − 1 colors in the neighborhood of v, and it is possible to color a vertex adjacent to v with the last color, so that all k colors would then appear in the neighborhood of v. The threat to the vertex v is said to be blocked or dealt if v is subsequently assigned a color, or it is no longer possible for v to have all k colors in its neighborhood. If two vertices x and y are under threat at atime then Alice can not block the threats on both these vertices and Bob wins the game
for a set of k colors because on one hand if Alice blocks the threat on x then Bob plays kth color in the unique uncolored neighborhood of y, on the other hand if Alice blocks the threat on y then Bob plays kth color in the unique uncolored neighborhood of x.
In this article, we have extended the study on game chromatic num- ber of families of double generalized petersen graphs DP (n, 1), DP (n, 2), DP (n, 3) and DP (n, m).

2. Definite values of χg(DP (n, m)), m ∈ {1, 2, 3}

The double generalized petersen graph is the generalization of general- ized petersen graph. It is also three regular graph like generalized petersen graph. In the double generalized petersen graph, there are two cycle, one is called outer cycle and other is called inner cycle. Each vertex of the both cycles is attached to a pendent vertex, the pendent vertices that are attached to outer cycle are called outer pendent vertices and the pendent vertices that are attached to inner cycle are called inner pendent vertices. The double generalized petersen graph DP (n, m) is obtained byattaching the vertices of outer pendent vertices to inner pendent vertices lying at dis- tance m. The length of the outer and inner cycle is n, thus the number of vertices are 4n and the number of edges in the DP (n, m) are 6n. DP (n, m), n ≥ 3 and m ∈ Zn − {0}, 2 ≤ 2m < n, has vertex set {xi, yi, ui, vi|i ∈ Zn}, edge set {xixi+1, yiyi+1, uivi+m, viui+m, xiui, yivi|i ∈ Zn}.

Lemma 1. $\chi_g(DP(3, 1)) = 4$.



Proof.

Fig. 1. DP (3, 1)

The double generalized petersen graph DP (3, 1) with labeled vertices is shown in fig. 1. It is accessible that χg(DP (3, 1)) > 2.

Bob's winning strategy for 3 colors:

Let the set of useable colors be {1, 2, 3}, then there are two cases for Alice to play her first turn. Case 1: If Alice plays color 1 in vertex xi (or yi), where $0 \leq i \leq 2$, then Bob responds by playingcolor 1 in the vertex yi (or xi). Then Alice has the following three choices to play her secondturn.

Subcase 1.1: If Alice plays color 1 in the vertex ui−1 (or ui+1) in her second turn then Bob plays color 2 in the vertex vi+1 in his second turn. In this way two vertices ui and yi+1 are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 1.2: If Alice plays color 1 in the vertex vi+1 (or vi−1) in her second turn then Bob playscolor 2 in the vertex ui−1 in his second turn. In this way two vertices xi−1 and vi are under threatat a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 1.3: In her second turn, if Alice plays some proper color in any other vertex of DP (3, 1) not described above then Bob plays according to the strategies of subcases 1.1 or 1.2 and winsthe game.

Case 2: If Alice plays color 1 in vertex ui (or vi), where $0 \leq i \leq 2$, then Bob responds by playing color 1 in the vertex vi (or ui). Then Alice has the following three choices to play her second turn.

Subcase 2.1: If Alice plays color 1 in the vertex xi−1 (or xi+1) in her second turn then Bob playscolor 2 in the vertex yi+1 in his second turn. In this way two vertices yi and vi+1 are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 2.2: If Alice plays color 1 in the vertex yi+1 (or yi−1) in her sec- ond turn then Bobplays color 2 in the vertex xi−1 in his second turn. In this way two vertices xi and ui−1 are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 2.3: In her second turn, if Alice plays some proper color in any other vertex of DP (3, 1) not described above then Bob plays according to the strategies of subcases 2.1 or 2.2 and winsthe game.

From above discussion it implies that the game chromatic number of
DP (3, 1) is at least 4. Thus $\chi g(DP\ (3, 1)) \geq 4$. Since maximum degree of
DP (3, 1) is 3, from equation 1 we have $\chi g(DP\ (3, 1)) \leq 4$.
Hence $\chi g(DP\ (3, 1)) = 4$.

Theorem 1. $\chi_g(DP\ (n,\ 1)) = 4$ *for all* $n > 3$.

Proof.

The double generalized petersen graph DP (n, 1) for n = 5 with labeled vertices is shown in fig.2.
It is accessible that $\chi_g(DP\,(n, 1)) > 2$.

Bob's winning strategy for 3 colors:
Let the set of useable colors be {1, 2, 3}, then there are two cases for Alice to play her first turn. Case 1: If Alice plays color 1 in vertex xi (or yi), where $0 \leq i \leq n - 1$, then Bob responds by playing color 1 in the vertex yi (or xi). Then Alice has the following three choices to play hersecond turn.
Subcase 1.1: If Alice plays color 1 in the vertex ui−1 in her second turn then Bob plays color 2 inthe vertex vi+1 in his second turn.

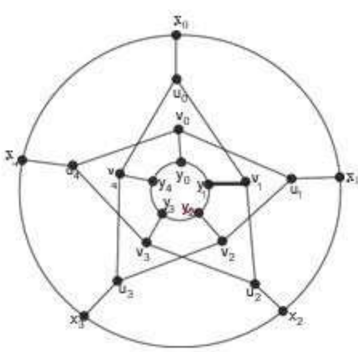


Fig. 2. DP (5, 1)

In this way two vertices ui and yi+1 are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.
Subcase 1.2: If Alice plays color 1 in the vertex vi+1 in her second turn then Bob plays color 2 in the vertex ui−1 in his second turn. In this way two vertices xi−1 and vi are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.
Subcase 1.3: In her second turn, if Alice plays some proper color in any other vertex of DP (n, 1) not described above then Bob plays according to the strategies of subcases 1.1 or 1.2 and winsthe game.
Case 2: If Alice plays color 1 in vertex ui (or vi), where $0 \leq i \leq n - 1$, then Bob responds by playing color 1 in the vertex vi (or ui). Then Alice has the following three choices to play her second turn.
Subcase 2.1: If Alice plays color 1 in the vertex xi−1 in her second turn then Bob plays color 2 in the vertex yi+1 in his second turn. In this way two vertices yi and vi+1 are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 2.2: If Alice plays color 1 in the vertex yi+1 in her second turn then Bob plays color 2 in the vertex xi−1 in his second turn. In this way two vertices xi and ui−1 are under threat at a time. Alice can not block the threats on both theses vertices

From above discussion it implies that the game chromatic number of DP $(n,$ 1) is at least 4. Thus $\chi_g(DP\ (n,\ 1)) \geq 4$. Since

and Bob wins the game.

Subcase 2.3: In her second turn, if Alice plays some proper color in any other vertex of DP (n, 1) not described above then Bob plays according to the strategies of subcases 2.1 or 2.2 and winsthe game.

maximum degree of DP $(n,$ 1) is 3, from equation 1 we have $\chi_g(DP\ (n,\ 1)) \leq 4$.

Hence $\chi_g(DP\ (n, 1)) = 4$ for $n > 3$.

Theorem 2. $\chi_g(DP\ (n, 2)) = 4$ *for all* $n > 4$

Proof.

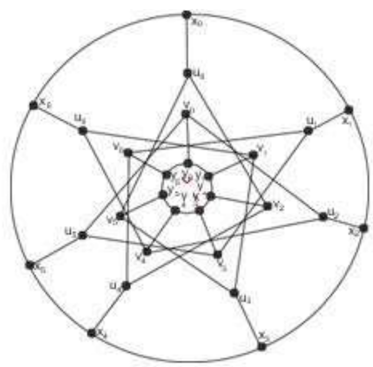


Fig. 3. DP (7, 2)

The double generalized petersen graph DP (n, 2) for n = 7 with labeled vertices is shown in fig. 3.

It is accessible that χg(DP (n, 2)) > 2.

Bob's winning strategy for 3 colors:

Let the set of useable colors be {1, 2, 3}, then there are two cases for Alice to play her first turn.

Case 1: If Alice plays color 1 in vertex xi (or yi+1), where 0 ≤ i ≤ n − 1, then Bob responds by playing color 1 in the vertex yi+1 (or xi). Then Alice has the following three choices to play her second turn.

Subcase 1.1: If Alice plays color 1 in the vertex ui−1 in her second turn then Bob plays color 2 in the vertex vi+2 in his second turn. In this way two vertices ui and yi+2 are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 1.2: If Alice plays color 1 in the vertex vi+2 in her second turn then Bob plays color 2 in the vertex ui−1 in his second turn. In this way two vertices xi−1 and vi+1 are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 1.3: In her second turn, if Alice plays some proper color in any other vertex of DP (n, 2) not described above then Bob plays according to the strategies of subcases 1.1 or 1.2 and wins the game.

Case 2: If Alice plays color 1 in vertex u_i (or v_{i+1}), where $0 \leq i \leq n - 1$, then Bob responds by playing color 1 in the vertex v_{i+1} (or u_i). Then Alice has the following three choices to play her second turn.

Subcase 2.1: If Alice plays color 1 in the vertex x_{i-1} in her second turn then Bob plays color 2 in the vertex y_{i+2} in his second turn. In this way two vertices y_{i+1} and v_{i+2} are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 2.2: If Alice plays color 1 in the vertex y_{i+2} in her second turn then Bob plays color 2 in the vertex x_{i-1} in his second turn. In this way two vertices x_i and u_{i-1} are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 2.3: In her second turn, if Alice plays some proper color in any other vertex of DP (n, 2) not described above then Bob plays according to the strategies of subcases 2.1 or 2.2 and wins the game.

From above discussion it implies that the game chromatic number of DP (n, 2) is at least 4. Thus $\chi_g(DP(n, 2)) \geq 4$. Since maximum degree of DP (n, 2) is 3, from equation 1 we have $\chi_g(DP(n, 2)) \leq 4$.

Hence $\chi_g(DP(n, 2)) = 4$ for $n > 4$.

Theorem 3. $\chi_g(DP(n, 3)) = 4$ *for all* $n > 6$

Proof.

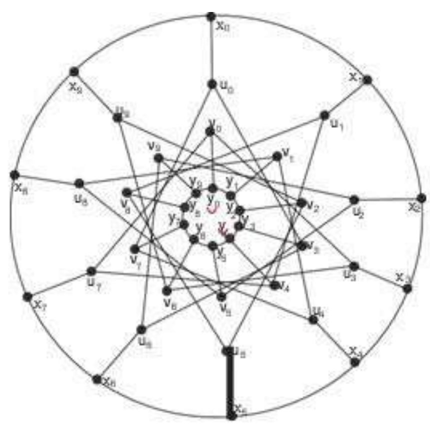


Fig. 4. DP (10, 3)

The double generalized petersen graph DP (n, 3) for n = 10 with labeled vertices is shown in fig. 4.

It is accessible that $\chi_g(DP(n, 3)) > 2$.

Bob's winning strategy for 3 colors:

Let the set of useable colors be {1, 2, 3}, then there are two cases for Alice to play her first turn.

Case 1: If Alice plays color 1 in vertex x_i (or y_{i+2}), where $0 \leq i \leq n - 1$, then Bob responds by playing color 1 in the vertex y_{i+2} (or x_i). Then Alice has the following three choices to play her second turn.

Subcase 1.1: If Alice plays color 1 in the vertex ui−1 in her second turn then Bob plays color 2 in the vertex vi+3 in his second turn. In this way two vertices ui and yi+3 are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 1.2: If Alice plays color 1 in the vertex vi+3 in her second turn then Bob plays color 2 in the vertex ui−1 in his second turn. In this way two vertices xi−1 and vi+2 are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 1.3: In her second turn, if Alice plays some proper color in any other vertex of DP (n, 2) not described above then Bob plays according to the strategies of subcases 1.1 or 1.2 and winsthe game.

Case 2: If Alice plays color 1 in vertex ui (or vi+2), where $0 \leq i \leq n - 1$, then Bob responds by playing color 1 in the vertex vi+2 (or ui). Then Alice has the following three choices to play her second turn.

Subcase 2.1: If Alice plays color 1 in the vertex xi−1 in her second turn then Bob plays color 2 in the vertex yi+3 in his second turn. In this way two vertices yi+2 and vi+3 are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 2.2: If Alice plays color 1 in the vertex yi+3 in her second turn then Bob plays color 2 in the vertex xi−1 in his second turn. In this way two vertices xi and ui−1 are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 2.3: In her second turn, if Alice plays some proper color in any other vertex of DP (n, 2) not described above then Bob plays according to the strategies of subcases 2.1 or 2.2 and winsthe game.

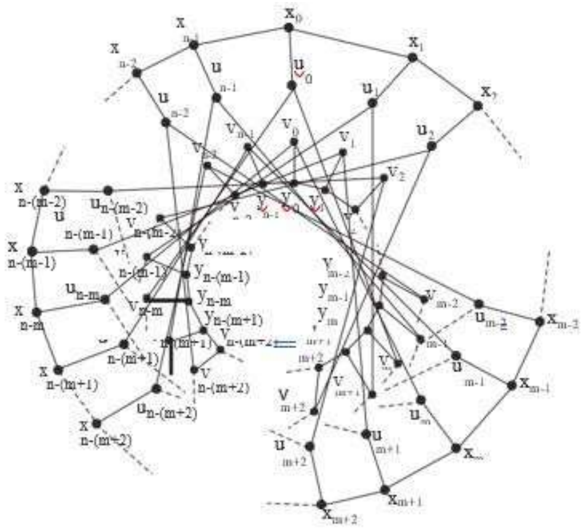
From above discussion it implies that the game chromatic number of DP (n, 3) is at least 4. Thus $\chi g(DP\ (n,\ 3)) \geq 4$. Since maximum degree of DP (n, 3) is 3, from equation 1 we have $\chi g(DP(n, 3)) \leq 4$.

Hence $\chi g(DP\ (n, 3)) = 4$ for $n > 6$.

3. Generalization of result for DP (n, m), n > 2m

Now the game chromatic number is generalized for double generalized pe- tersen graph DP (n, m) for $n > 2m$.

Theorem 4. $\chi g(DP\ (n, m)) = 4$ for all $n > 2m$ and $m \geq 4$.



Proof.

Fig. 5. DP (n, m), n > 2m

The double generalized petersen graph DP (n, m) with labeled vertices is shown in fig. 5. It is accessible that χg(DP (n, m)) > 2.

Bob's winning strategy for 3 colors:

Let the set of useable colors be {1, 2, 3}, then there are two cases for Alice to play her first turn. Case 1: If Alice plays color 1 in vertex xi (or yi+(m−1)), where 0 ≤ i ≤ n−1, then Bob responds by playing color 1 in the vertex yi+(m−1) (or xi). Then Alice has the following three choices to play her second turn.

Subcase 1.1: If Alice plays color 1 in the vertex ui−1 in her second turn then Bob plays color 2 in the vertex vi+m in his second turn. In this way two vertices ui and yi+m are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 1.2: If Alice plays color 1 in the vertex vi+m in her second turn then Bob plays color 2 in the vertex ui−1 in his second turn. In this way two vertices xi−1 and vi+(m−1) are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 1.3: In her second turn, if Alice plays some proper color in any other vertex of DP (n, m)not described above then Bob plays according to the strategies of subcases 1.1 or 1.2 and winsthe game.

Case 2: If Alice plays color 1 in vertex ui (or vi+(m−1)), where 0 ≤ i ≤ n−1, then Bob responds by playing color 1 in the vertex vi+(m−1) (or ui). Then Alice has the following three choices to play her second turn.

Subcase 2.1: If Alice plays color 1 in the vertex xi−1 in her second turn then Bob plays color 2 in the vertex yi+m in his second turn. In this way two vertices yi+(m−1) and vi+m are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 2.2: If Alice plays color 1 in the vertex yi+m in her second turn then Bob plays color 2 in the vertex xi−1 in his second turn. In this way two vertices xi and ui−1 are under threat at a time. Alice can not block the threats on both theses vertices and Bob wins the game.

Subcase 2.3: In her second turn, if Alice plays some proper color in any other vertex of DP (n, 2) not described above then Bob plays according to the strategies of subcases 2.1 or 2.2 and wins the game.

From above discussion it implies that the game chromatic number of DP (n, m) is at least 4. Thus χg(DP (n, m)) ≥ 4. Since maximum degree of DP (n, m) is 3, from equation 1 we have χg(DP (n, m)) ≤ 4.

Hence χg(DP (n, m)) = 4 for n > 2m.

The following statement is an immediate consequence of the previous results.

Corollary 1. χg(DP (n, m)) = 4 for all n > 2m and m ∈ N.

References

1. M.S. Akhtar, Usman Ali, Ghulam Abbas, Mutahira Batool, On the game chromatic number of splitting graphs of path and cycle, Theoret. Comput. Sci., http://doi.org/10.1016/j.tcs.2019.05.035, (2019).
2. S.A. Bokhary and Tanveer Iqbal, Game chromatic number of cartesian and corona product graphs, J. Algebra Comb. Discrete Appl., 5(3)(2018), 129-136.
3. S.A. Bokhary and M.S. Akhtar, Game chromatic number of some convex polytope graphs, Utilitas Mathematica, 104 (2017), 15-22.
4. T. Bartnicki, B. Bresˇar, J. Grytczuk, M. Kovsˇe, Z. Miechowicz, I. Peterin, Game chromatic number of Cartesian product graphs, Electron. J. Combin., 15 (2008), R72, 13pp.
5. H. L. Bodlaender, On the complexity of some coloring games, Internat. J. Found. Com- put. Sci., 2 (1991) 133-147.
6. U. Faigle, U. Kern, H. Kierstead, W. T. Trotter, On the game chromatic number of some classes of graphs, Ars Combin., 35 (1993), 143-150.
7. D. J. Guan, X. Zhu, Game chromatic number of outerplanar graphs, J. Graph Theory, 30 (1999), 67-70.
8. Imran Javed, Shabbir Ahmad, M. Naeem Azhar, On the Metric Dimension of Generalized Peterson Graphs, Ars Combinatoria-Waterloo then Winnipeg, July 2012.
9. H. A. Kierstead, T.Trotter, Planar graph coloring with uncooperative part- ner, J. Graph Theory, 30 (1990), 67-70.
10. C. Sia, The game chromatic number of some families of cartesin product graphs, AKCE Int.J.Graphs Comb., 6 (2009), 315-327.
11. Shaheen, Ramy, Ziad Kanaya and Khaled Alshehada, Game Chromatic Num- ber of Generalized Petersen Graphs and Jahangir Graphs, Journal of Applied Mathematics 2020(2020).
12. X. Zhu, Game coloring the cartesian product of graphs, J. Graph Theory, 59 (2008), 261-278.